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جامعة ستاردوم

مجلة ستاردوم العلمية للدراسات الطبيعية والهندسية

مجلة ستاردوم العلمية للدراسات الطبيعية والهندسية

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A NEW EXTENDED BETA FUNCTION INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION AND IT'S APPLICATIONS

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Abstract

The main object of this paper is to introduce a new extension of the beta function involving the generalized Mittag-leffler function and study its important properties, like integral representation, summation formula, derivative formula, beta distribution and transform formula. We introduce new extended hypergeometric and confluent hypergeometric functions.

Keywords: Beta function, Beta Distribution, Confluent hypergeometric function, Gamma function, Hypergeometric function, Summation formulas, Transform formula

Introduction:

There are many extensions and generalizations of the beta function, hypergeometric function and confluent hypergeometric function have been considered by several authors (see [2, 3, 4, 5, 6, 7, 10, 11, 12]). In this paper, we study another extension of the Euler Beta function and investigate various formulas, such as integral representation, summation formula and derivative formula. Further, we obtain beta distribution and its some statistical formulas. We extend also the definition of hypergeometric and confluent hypergeometric functions and study their various properties.

The classical Gauss hypergeometric function (see [1]) is defined as

$$F(\delta_1, \delta_2; \delta_3; \tau) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n (\delta_2)_n}{(\delta_3)_n} \frac{\tau^n}{n!}, \tag{1.1}$$

where $(\delta)_n$ $(\delta \in \mathbb{C})$ is the Pochhammer symbol defined by

$$(\delta)_n = \frac{\Gamma(\delta + n)}{\Gamma(\delta)} \,. \tag{1.2}$$

The confluent hypergeometric function (see [1]) is defined by

$$\Phi(\delta_1; \delta_2; \tau) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n}{(\delta_2)_n} \frac{\tau^n}{n!} \,. \tag{1.3}$$

The Gamma function $\Gamma(\tau)$ developed by Euler [1] with the intent to extend the factorials to values between the integers is defined by the definite integral

$$\Gamma(z) = \int_0^1 e^{-t}\, t^{z-1}\, dt \quad , \quad Re(z) > 0\,. \tag{1.4}$$

Among various extensions of gamma function, we mention here the extended gamma function [2] defined by Chaudhry and Zubair

$$\Gamma_p(z) = \int_0^1 t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt \quad , \quad (Re(p) > 0). \tag{1.5}$$

The Euler Beta function $B(z_1, z_2)$ (see [1]) is defined by

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}\, (1-t)^{z_2-1}\, dt \tag{1.6}$$

$$= \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = \frac{(z_1-1)!\ (z_2-1)!}{(z_1+z_2-1)!}, \tag{1.7}$$

where $z! = \Gamma(z+1)$, $z = 0,1,2,4, \ldots$, $(Re(z_1) > 0 \ , Re(z_2) > 0)$.

In 1997, Choudhary et al. [3] introduced an extension of the beta function defined by

$$B^p(z_1, z_2) = \int_0^1 t^{z_1-1}\, (1-t)^{z_2-1}\, exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.8}$$

where $Re(p) \geq 0$, $(Re(z_1) > 0 \ , Re(z_2) > 0)$.

Chaudhary et al. [4] used the new extended the beta function $B^p(\delta_1, \delta_2)$ to introduce an extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F^p(\delta_1, \delta_2, \delta_3; \tau)$$

$$= \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^p(\delta_1+n, \delta_3-\delta_2)}{B(\delta_2, \delta_3-\delta_2)} \frac{\tau^n}{n!}\,, \tag{1.9}$$

$(p \geq 0\,,\ |\tau| < 1\,,\ \ Re(\delta_3) > Re(\delta_2) > 0),$

and

$$\Phi^{p}(\delta_1;\delta_2;\tau)=\sum_{n=0}^{\infty}\frac{B^{p}(\delta_1+n,\delta_2-\delta_1)}{B(\delta_1,\delta_2-\delta_1)}\ \frac{\tau^n}{n!}, \tag{1.10}$$

$$(p\geq 0\,,\quad Re(\delta_2)>Re(\delta_1)>0).$$

In 2018, Shadab et al. [12] introduced an extended the beta function in terms of the classical Mittag-Leffler function defined as

$$B_{\alpha}^{p}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha}exp\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.11}$$

$$Re(p)\geq 0\,, Re(\delta_1)>0\,, Re(\delta_2)>0,\qquad \alpha\in\mathbb{R}_0^{+},$$

where $E_{\alpha}(\cdot)$ is the classical Mittag-Leffler function defined as [9]

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{\tau^n}{\Gamma(\alpha n+1)}\,, \tag{1.12}$$

where $\qquad x\in C\ \ ,\alpha\in\mathbb{R}_0^{+}.$

Shadab et al. [12] used the extended Beta function to introduce a new extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F_{\alpha,\beta}(\delta_1,\delta_2,\delta_3;\tau)=\sum_{n=0}^{\infty}(\delta_1)_n\ \frac{B_{\alpha}^{p}(\delta_1+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\ \frac{\tau^n}{n!}. \tag{1.13}$$

$$(p\in\mathbb{R}_0^{+},\ \ ,\ \ \alpha\in\mathbb{R}^{+}\,,\ |\tau|<1,\ Re(\delta_1)>0\,,\ Re(\delta_3)>Re(\delta_2)>0).$$

The confluent hypergeometric function is defined as Φ

$$\Phi_{\alpha,\beta}(\delta_1;\delta_2;\tau)=\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}(\delta_1+n,\delta_2-\delta_1)}{B(\delta_1,\delta_2-\delta_1)}\ \frac{\tau^n}{n!}\,, \tag{1.14}$$

$$(p\in\mathbb{R}_0^{+},\ \ ,\ \ \alpha\in\mathbb{R}^{+}\,,\qquad |\tau|<1,\ Re(\delta_2)>Re(\delta_1)>0).$$

In 2022, Khan et al. [7] introduced a new extended Beta function in terms of the classical Mittag-Leffler function defined as

$$B_{\alpha,\beta}^{p,\mu,v}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}\left(-\frac{p}{t^u(1-t)^v}\right)dt, \qquad (1.15)$$

$$Re(p)>0, \qquad Re(\delta_1)>0\,, Re(\delta_2)>0, \qquad \alpha,\beta\in\mathbb{R}_0^+\,,\mu,v\in\mathbb{R}^+.$$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined as [12]

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)}\,, \qquad (1.16)$$

where $x\in\mathbb{C}\ ,\alpha,\beta\in\mathbb{R}_0^+$.

2. A new extension of the beta function

In this section, we introduce a new extension of the extended Beta function $B_{\alpha,\beta}^{p,\mu,v}(x,y)$ and investigate various properties and representations

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt, \qquad (2.1)$$

$$Re(p)>0, Re(\delta_1)>0\,, Re(\delta_2)>0, \qquad \alpha,\beta,\gamma,\sigma\in\mathbb{R}_0^+\,,\mu,v\in\mathbb{R}^+.$$

where $E_{\alpha,\beta}^{\gamma,\sigma}\,(.)$ is the generalized Mittag-Leffler function defined as [13]

$$E_{\alpha,\beta}^{\gamma,\rho}\,(x)=\sum_{k=0}^{\infty}\frac{(\gamma)_{\rho k}}{\Gamma(\alpha k+\beta)}\,\frac{x^k}{k!} \qquad (2.2)$$

where $\alpha,\beta,\gamma\,\epsilon\mathbb{C},\ Re\,(\alpha)\,,Re\,(\beta)\,,Re\,(\gamma)\ >0,\ \ \rho\ \in\ (0,1)\ \cup\ \mathbb{N}.$

If $\sigma=1$, in Eq. (2.1), we get

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,1)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t^u(1-t)^v}\right)dt. \qquad (2.3)$$

If $\sigma=\gamma=1$, in Eq. (2.1), we get

$$B_{\alpha,\beta}^{(p,\mu,v,1,1)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}\left(-\frac{p}{t^u(1-t)^v}\right)dt. \qquad (2.4)$$

If $\gamma=\sigma=\alpha=\beta=u=v=1$, in Eq. (2.1), we get

$$B_{1,1}^{(p,1,1,1,1)}(\delta_1,\delta_2)=B^p(\delta_1,\delta_2)=B(\delta_1,\delta_2;p). \tag{2.5}$$

If $\gamma=\sigma=u=v=1$, in Eq. (2.1), we get

$$B_{\alpha,\beta}^{(p,1,1,1,1)}(\delta_1,\delta_2)=B_{\alpha,\beta}^p(\delta_1,\delta_2)=B_{\alpha,\beta}(\delta_1,\delta_2;p). \tag{2.6}$$

If $\gamma=\sigma=\beta=u=v=1$, in Eq. (2.1), we get

$$B_{\alpha,1}^{(p,1,1,1,1)}(\delta_1,\delta_2)=B_{\alpha}^p(\delta_1,\delta_2)=B_{\alpha}(\delta_1,\delta_2;p). \tag{2.7}$$

3. Properties of $\boldsymbol{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)}$

In this section we obtain some interesting relation of summation formulas for $B_{\alpha,\beta}^{(p,\gamma,\mu,v)}(\delta_1,\delta_2)$

Theorem 3.1. The following integral representations hold

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$$

$$=2\int_0^{\frac{\pi}{2}}\cos^{2\delta_1-1}\theta\,\sin^{2\delta_2-1}\theta\,E_{\alpha,\beta}^{\gamma,\sigma}(-p(\sec^2\theta)^{\mu}(cosec^2\,\theta)^{v})d\theta, \tag{3.1}$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$$

$$=\int_0^{\infty}\frac{u^{\delta_1-1}}{(1+u)^{\delta_1+\delta_2}}\,E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(1+u)^{\mu+v}}{u^{\mu}}\right)du, \tag{3.2}$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=2^{1-\delta_1-y}\int_{-1}^{1}(1-u)^{\delta_1-1}(1-u)^{\delta_2-1}$$

$$\times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right)du\,, \tag{3.3}$$

$$Re(p)>0, Re(\delta_1)>0\,, Re(\delta_2)>0,\qquad \alpha,\beta,\gamma,\sigma\in\mathbb{R}^+\,,\mu,v\in\mathbb{R}^+.$$

Proof. Let $t = \cos^2\theta$, $t = \frac{u}{1+u}$, $t = \frac{1+u}{2}$, respectively in equation (2.1), we obtain the above representations.

Remark 3.1. If we take $\gamma = \sigma = 1$, in the integral representation of (3.1), Theorem (3.1), we obtain corresponding integrals for $B_{\alpha,\beta}^{(p,\mu,v)}(\delta_1,\delta_2)$ in [7].

If we take $\gamma = 1$, $\sigma = 1$, $\alpha = 1$, $\beta = 1$, $\mu = 1$, $v = 1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B(\delta_1,\delta_2;p)$ in [3].

If we take $\gamma = 1$, $\sigma = 1$, $\beta = 1$, $\mu = 1$, $v = 1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B_\alpha(\delta_1,\delta_2;p)$ in [12].

Theorem 3.2. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2) = \sum_{k=0}^{n}\binom{n}{k} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+k,\delta_2+n-k), n \in N_0. \quad (3.4)$$

Proof. We find from (2.1) that

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2) = \int_0^1 t^{\delta_1-1}(1-t)^{\delta_2-1}[t+(1-t)]E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt$$

$$= B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+1,\delta_2) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2+1). \quad (3.5)$$

Repeating the same argument to the above two terms in (3.5), we obtain

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2) = B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+2,\delta_2)$$

$$+2B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+1,\delta_2+1) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2+1). \quad (3.6)$$

Continuing this process, by using mathematical induction we get the desired result (3.4).

Theorem 3.3. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2) = \sum_{k=0}^{n}\frac{(\delta_2)_n}{n!} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,1) \quad n \in N_0 \quad (3.7)$$

$$Re(p) > 0, Re(\delta_1) > 0\,, Re(\delta_2) > 0, \qquad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+\,, \mu, v \in \mathbb{R}^+.$$

Proof. To prove the above result, by using the generalized binomial theorem defined as

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!} \qquad (|t| < 1). \tag{3.8}$$

We fined

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \int_0^1 \sum_{n=0}^{\infty} (\delta_2)_n \frac{t^{\delta_1 n-1}}{n!}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt. \tag{3.9}$$

Interchanging the order of integral and summation in the above equation and using (2.1), we get the desired result (3.7).

Theorem 3.4. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{k=0}^{n} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 1, \delta_2 + 1) \tag{3.10}$$

$$Re(p) > 0\,, Re(\delta_1) > 0\,, Re(\delta_2) > 0, \qquad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+\,, \mu, v \in \mathbb{R}^+.$$

Proof. Using the relation

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n \qquad (|t| < 1) \tag{3.11}$$

in (2.1), we obtain

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \int_0^1 (1-t)^{\delta_2} \sum_{n=0}^{\infty} t^{n+\delta_1-1}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt,$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{n=0}^{\infty} \int_0^1 (1-t)^{\delta_2}\, t^{n+\delta_1-1}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt,$$

which in view of (2.1), we get the desired result (3.10).

Remark 3.2. In case $\gamma = 1, \sigma = 1, \alpha = 1, \beta = 1, \mu = 1, v = 1$ of (3.4) for $n = 1$, (3.7) and (3.10) reduces to corresponding results in [3].

In case $\gamma = 1, \sigma = 1, \beta = 1, \mu = 1, v = 1$ of (3.4) for $n = 1$, (3.7) and (3.10) reduces to corresponding results in [12].

In case $\gamma = 1, \sigma = 1$, of (3.4) for $n = 1$, (3.7) and (3.10) reduces to corresp-onding results in [7].

In case $\sigma = 1$ of (3.4) for $n = 1$, (3.7) and (3.10) ,we get the following new results

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2) = \sum_{k=0}^{n} \frac{(\delta_2)_n}{n!} B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1 + n, 1) \quad n \in N_0, \tag{3.12}$$

and

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2) = \sum_{k=0}^{n} B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1 + 1, \delta_2 + 1). \tag{3.13}$$

4. Beta distribution of $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$

We now define the beta distribution of (2.1), and obtain its mean, variance, moment generating function and cumulative distribution.

For $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$, the Beta distribution is given by

$$f(t) = \begin{cases} \frac{1}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)} t^{\delta_1 - 1} (1-t)^{\delta_2 - 1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) & (0 < t < 1), \\ 0, & otherwise \end{cases} \tag{4.1}$$

$$\delta_1, \delta_2 \epsilon \mathbb{R}, \quad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+, \quad \mu, v \in \mathbb{R}^+.$$

For $d \in R$, the d^{th} moment of a random variable X defined as

$$\rho = E(X^d) = \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + d, \ \delta_2)}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)}, \tag{4.2}$$

$$\delta_1, \delta_2 \in \mathbb{R}, \ p \geq 0, \ \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+, \quad \mu, v \in \mathbb{R}^+.$$

The variance of the distribution is defined by

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\ \delta_2) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+2,\delta_2) - \left\{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+1,\delta_2)\right\}^2}{\left\{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)\right\}^2}. \quad (4.3)$$

The moment generating function of the distribution is defined as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, E(X^n) = \frac{1}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)} \sum_{n=0}^{\infty} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_2)\, \frac{t^n}{n!}. \quad (4.4)$$

The cumulative distribution is defined as

$$f(z) = \frac{B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+d,\ \delta_2)}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)}. \quad (4.5)$$

where

$$B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2) = \int_0^z t^{\delta_1-1}\,(1-t)^{\delta_2-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt, \quad (4.6)$$

$$(p > 0, \quad -\infty < \mu, v < \infty),$$

is the extended incomplete Beta function.

5-Generalization of extended hypergeometric and confluent hypergeometric functions

Here, we introduce a generalization of extended hypergeometric and confluent hypergeometric functions in terms of $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$

The extended hypergeometric function is defined as

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau) = \sum_{n=0}^{\infty} (\delta_1)_n\, \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\, \frac{\tau^n}{n!}, \quad (5.1)$$

$$(p \geq 0\,,\ \ |\tau| < 1\,,\ \ \alpha,\beta,\gamma,\sigma,\mu,v > 0\,,\ \ Re(\delta_3) > Re(\delta_2) > 0).$$

The confluent hypergeometric function is defined as

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\ \frac{\tau^n}{n!}, \tag{5.2}$$

$$(p\geq 0\,,\ \ \alpha,\beta,\gamma,\sigma,\mu,v>0\,,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

Remark 5.1. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$ in (5.1) and (5.2), we obtain corresponding results in [4].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (5.1) and (5.2), we obtain corresponding results in [12].

In case $\sigma=\gamma=1$ in (5.1) and (5.2), we obtain corresponding result in [7].

In case $\sigma=1$ in (5.1) and (5.2), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau)=\sum_{n=0}^{\infty}(\delta_1)_n\ \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\ \frac{\tau^n}{n!}, \tag{5.3}$$

$$(p\geq 0\,,\ \ |\tau|<1\,,\ \ \alpha,\beta,\gamma,\mu,v>0\,,\ \ Re(\delta_3)>Re(\delta_2)>0)$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\ \frac{\tau^n}{n!}, \tag{5.4}$$

$$(p\geq 0\,,\ \ \alpha,\beta,\gamma,\mu,v>0\,,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

6. Integral Representation and derivative formula for extended Gauss hypergeometric functions

Theorem 6.1. The following integral representations for the extended hypergeometric function $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$ and confluent hypergeometric function $\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)$ holds

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-\tau t)^{-\delta_1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.1}$$

$(p\in\mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v\in\mathbb{R}^+\ ;\ and\ \ arg|1-\tau|<\pi,\ Re(\delta_3)>Re(\delta_2)>0).$

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}e^{zt}\,E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.2}$$

$(p\in\mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v\in\mathbb{R}^+\ ;\quad Re(\delta_3)>Re(\delta_2)>0).$

Proof. By using the definition of $B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y)$ in (2.1) into (5.1) and interchanging the order of integration and summation, which is verified under the condition here, we have

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}$$

$$\times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right)\sum_{n=0}^{\infty}(\delta_1)_n\frac{(\tau t)^n}{n!}. \tag{6.3}$$

Using the binomial theorem in (3.11) to the summation formula in (6.3), we get the desired result (6.1).

Similarly, we can obtain (6.2).

Remark 5.1. In case $\alpha=\beta\sigma=\gamma=\mu=v=1$ in (6.1) and (6.2), we obtain the corresponding result in [4].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (6.1) and (6.2), we obtain the corresponding result in [12].

In case $\sigma=\gamma=1$ in (6.1) and (6.2), we obtain the corresponding result in [7].

In case $\sigma = 1$ in (6.1) and (6.2), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-\tau t)^{-\delta_1}\, E_{\alpha,\beta}^{\gamma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.4}$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\mu,v \in \mathbb{R}^+;\ and\ \ arg|1-\tau| < \pi,\ Re(\delta_3) > Re(\delta_2) > 0),$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times \int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}e^{zt}\, E_{\alpha,\beta}^{\gamma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.5}$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\mu,v \in \mathbb{R}^+;\quad Re(\delta_3) > Re(\delta_2) > 0).$$

Theorem 6.2. The following derivative formula for extended Gauss hypergeometric and confluent hypergeometric function holds:

$$\frac{d^n}{d\tau^n}\left\{F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)\right\} = \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}$$

$$\times F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_2+n;\delta_3+n;\tau), \tag{6.6}$$

and

$$\frac{d^n}{d\tau^n}\left\{\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)\right\} = \frac{(\delta_2)_n}{(\delta_3)_n}\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3+n;\tau), \tag{6.7}$$

where

$$(p \geq 0\,,\qquad \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+;\quad Re(\delta_3) > Re(\delta_2) > 0).$$

Proof. Differentiating (5.1) and (5.2) with respect to τ and using the following formula

$$B(\delta_2,\delta_3-\delta_2)=\frac{\delta_3}{\delta_2}\,B(\delta_2+1,\delta_3-\delta_2)\ \ and\ \ (\delta)_n=\delta(\delta+1)_n. \qquad (6.8)$$

we obtain the derivative formulas (6.6) and (6.7) for $n=1$. Easily applying the same process, we get the desired results (6.6) and (6.7).

Remark 6.2. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$ in (6.6) and (6.7), we obtain the corresponding result in [4].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (6.6) and (6.7), we obtain the correspon -ding result in [12].

In case $\sigma=\gamma=1$ in (6.6) and (6.7), we obtain the corresponding result in [7].

In case $\sigma=1$ in (6.6) and (6.7), we get the following new results

$$\frac{d^n}{d\tau^n}\left\{F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau)\right\}=\frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}$$
$$\times F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1+n,\delta_2+n;\delta_3+n;\tau), \qquad (6.9)$$

and

$$\frac{d^n}{d\tau^n}\left\{\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau)\right\}=\frac{(\delta_2)_n}{(\delta_3)_n}\ \Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2+n,\delta_3+n;\tau). \qquad (6.10)$$

7. Transformation and summation formulas

Theorem 7.1. The following formulas for the extended hypergeometric and confluent hypergeometric function hold

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=(1-\tau)^{-k}\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1-\tau}\right), \qquad (7.1)$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;1-\frac{1}{\tau}\right)=\tau^k\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;1-\tau), \qquad (7.2)$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1+\tau}\right)=(1+\tau)^k\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;-\tau), \qquad (7.3)$$

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)=e^{\tau}\,\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_3-\delta_2;\delta_3;-\tau), \qquad (7.4)$$

$$(p\in\mathbb{R}_0^+,\ \ |\tau|<1,\ \ \alpha,\beta,\gamma,\sigma,\mu,v>0,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

Proof. Replacing t by $1-t$ and substituting

$$\left(1-\tau(1-t)\right)^{-\delta_1}=(1-\tau)^{-\delta_1}\left(1+\frac{\tau}{1-\tau}\,t\right)^{-\delta_1},$$

in (6.1), we obtain

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=\frac{(1-\tau)^{-\delta_1}}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times\int_{-1}^{1}t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\left(1+\frac{\tau}{1-\tau}\,t\right)^{-\delta_1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \qquad (7.5)$$

$$=\frac{(1-\tau)^{-\delta_1}}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times\int_{-1}^{1}t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\left(1-\frac{-\tau}{1-\tau}\,t\right)^{-\delta_1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right). \qquad (7.6)$$

In view of (6.1), we get the desired result (7.1).

Replacing τ by $1-\frac{1}{\tau}$ and $\frac{\tau}{1+\tau}$ in (7.1) yield (7.2) and (7.3) respectively.

Similarly as (7.1), we can establish (7.4).

Remark 7.1. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$ in (7.1) and (7.4), we obtain the corresponding result in [4].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (7.1) to (7.4), we obtain the corresponding result in [12].

In case $\sigma=\gamma=1$ in (7.1) to (7.4), we obtain the corresponding result in [7].

In case $\sigma=1$ in (7.1) to (7.4), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau)=(1-\tau)^{-k}\,F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1-\tau}\right), \qquad (7.7)$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;1-\frac{1}{\tau}\right)=\tau^{k}\,F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;1-\tau), \qquad (7.8)$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1+\tau}\right)=(1+\tau)^k\,F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;-\tau), \tag{7.9}$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau)=e^{\tau}\,\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_3-\delta_2;\delta_3;-\tau), \tag{7.10}$$

$$(p\in\mathbb{R}_0^+,\ \ |\tau|<1,\ \ \alpha,\beta,\gamma,\mu,v>0,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

Theorem 7.2. The following summation formula hold

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;1)=\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2,\delta_3-\delta_1-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}, \tag{7.11}$$

$$(p\in\mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v\in\mathbb{R}^+;\quad Re(\delta_3-\delta_1-\delta_2)>0).$$

Proof. Putting $\tau=1$ in (6.1) and using the definition (2.1), we obtain the desir-ed result (7.11).

Remark 7.2. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$, with $p=0$ in (7.11), we obtain the Gauss summation formula Γ for ${}_2F_1$

$${}_2F_1(\delta_1,\delta_2,\delta_3;1)=\frac{\Gamma(\delta_3)\,\Gamma(\delta_3-\delta_1-\delta_2)}{\Gamma(\delta_3-\delta_1)\Gamma(\delta_3-\delta_2)},\ (Re(\delta_3-\delta_1-\delta_2)>0)\,. \tag{7.12}$$

8. A generating function for $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$

Theorem 8.1. The following generating function for $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$

hold

$$\sum_{n=k}^{\infty}(\delta_1)_n\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+k,\delta_2,\delta_3;\tau)\ \ \frac{\tau^k}{k!}$$

$$=(1-t)^{-\delta_1}F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1+k,\delta_2,\delta_3;\frac{\tau}{1-t}\right), \tag{8.1}$$

$$(p\in\mathbb{R}_0^+,\ \ |t|<1,\ \ \alpha,\beta,\gamma,\sigma,\mu,v\ \in\mathbb{R}^+).$$

Proof. Let Δ be the left hand side (L.H.S) of (8.1). From (5.1), we have

$$\Delta= \sum_{k=0}^{\infty}(\delta_1)_k \left(\sum_{n=0}^{\infty}\frac{(\delta_1+k)_n\, B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{\tau^n}{n!}\right)\frac{t^k}{k!} \tag{8.2}$$

$$=\sum_{k=0}^{\infty}(\delta_1)_k \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\left(\sum_{n=0}^{\infty}(\delta_1+k)_n\frac{t^k}{k!}\right)\frac{\tau^n}{n!}$$

$$=(1-t)^{-\delta_1}\sum_{k=0}^{\infty}(\delta_1)_k \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\left(\frac{\tau}{1-t}\right)^n\frac{1}{n!}. \tag{8.3}$$

Finally by using (5.1) in (8.3), we get the right side of (8.1).

Remark 8.1. In case $\alpha,\beta,\sigma,\gamma,\mu,v=1$ in (8.1), we obtain the corresponding result in [10].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (8.1), we obtain the corresponding result in [12].

In case $\sigma=\gamma=1$ in (8.1), we obtain the corresponding result in [7].

In case $\sigma=1$ in (8.1), we get the following new result

$$\sum_{n=k}^{\infty}(\delta_1)_n\, F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1+k,\delta_2,\delta_3;\tau)\frac{\tau^k}{k!}$$
$$=(1-t)^{-\delta_1}F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1+k,\delta_2,\delta_3;\frac{\tau}{1-t}\right). \tag{8.4}$$

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